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The benefits of education and of useful knowledge, generally diffused through a community, are essential to the preservation of a free government.

Sam Houston

Cultivated mind is the guardian genius of Democracy, and while guided and controlled by virtue, the noblest attribute of man. It is the only dictator that freemen acknowledge, and the only security which freemen desire.

Mirabeau B. Lamar

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Volume XV, Number 3

Edited by

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Adjunct Professor of Pure Mathematics

Teachers of mathematics in Texas are cordially invited to use this bulletin for the expression of their views. The editor assumes no responsibility for statements of facts or opinions in the articles.

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PIONEERS OF MATHEMATICS, ASTRONOMY, AND ASTROLOGY IN AMERICA

C. E. CASTAÑEDA

Latin-American Librarian, The University of Texas

Though a treatise on arithmetic was published in Mexico City as early as 1556, and though by 1649 two full and detailed books on this subject were available in print, still we do not find mathematics forming a part of the curriculum of the National University, founded in 1553. Nor do we find any traces of the teaching of mathematics as such in the various colleges established during the sixteenth and seventeenth centuries in Mexico. But regardless of this fact, the rudiments of mathematics were certainly being taught throughout this time in private schools and by private teachers, for accountants there were in large numbers, and the numerous engineering undertakings such as the draining of the valley of Mexico, the building of bridges, the erection of superb buildings, like the National Cathedral, and other similar projects clearly point out a knowledge of mathematics beyond the rudimentary principles of plain arithmetic.

Curious to find out when mathematics was first officially taught in the National University of Mexico, called during the colonial régime the Royal and Pontifical University, having been founded by a royal decree of Charles V and a Bull of the Pope, the writer began to look over old records and books several years ago. At first his efforts proved futile, but one fine day he discovered that the first professor to teach mathematics in Mexico, as far as the records show, was a Mercedery Friar who in 1660 held the chair of Astrology and Mathematics in the Department of Medicine. The school of medicine was not established as a part of the University until 1580. At first only one class was offered, but gradually other subjects were added and thus mathematics finally came into the august halls of the university as a handmaid of medicine and in the none-too-flattering company of that pseudo-scientific jane, astrology.

That Fr. Diego Rodriguez was held in high esteem is evident from the notation placed after his name in the official roll of the faculty. "He is the bookkeeper of the institution, distinguished in his profession, a bachelor of arts, and a person of high rank in his Order." For eight years he held this chair until the time of his death in 1668.

Nothing has been found concerning the nature of the course on astrology and mathematics which the good friar taught, but it appears from a brief account of his life and works that he wrote his own text and several other books on the subject during his lifetime. It should be remembered that in the seventeenth century it was customary for the professor to "dictate" his course, that is, to read from rather full notes, while the class religiously took down the words of the professor.

In the entry found in the *Chronicle* of the University recording the death of this first teacher of mathematics it is stated that he is worthy of memory for his learning, his virtue, and his faith. He was not only a prelate of his Order, but he was also Commander of the Convent of the Order of Mercy in Mexico City, the highest office open to him. He was well known also as an eloquent sacred orator. He was eminent as an astrologer and mathematician. For many years he prepared annual almanacs. The chronicler further declares that "He wrote several curious and detailed books on mathematics and astrology, all in his own hand. These were three good and large volumes, for I saw them, and they were sent to Spain for the purpose of printing them. I do not think they were printed, however, but were returned to the library of the Convent of his Order, where they now are."

Fr. Diego Rodriguez was succeeded by Fr. Ignacio Muñoz, who took possession of his chair in August, 1668, after his appointment had been duly confirmed by the viceroy, the Marquis of Mancera. Soon after taking charge of his class, it seems that Fr. Muñoz had to go to Spain on business for the Crown, whereupon the chair was declared vacant, but this was contested by Fr. Muñoz, who claimed that he still was the legal possessor of the chair and that the University could not proceed to the election of a successor. The matter was taken to the viceroy, who decided that since Fr. Muñoz had not complied with the requisites set by the constitution of the university and had failed to obtain the consent of the *Cluatro* (university faculty) for his absence, the chair was to be considered vacant and the university should proceed to hold a public *Oposicion* (contest) to select a new professor of astrology and mathematics.

It is very interesting to look back at the method used then to select a new professor. All candidates who desired to be considered filed applications with the secretary of the university. The faculty and the student body were duly notified of the day on which the candidates were to appear before them to show publicly their qualifications for the position. Each candidate was given an hour to lecture or demonstrate on the subject he was to teach, and the faculty and student body took a secret vote on the candidates. At the conclusion of the public trial or contest, the secretary counted the votes in favor of each of the contestants and the one receiving a majority was declared victor and appointed to the chair.

Thus in 1672, due to the unauthorized absence of Fr. Ignacio Muñoz, who had gone to Spain on business for the Crown, a public *Oposicion* was held, but there being only one candidate it was assigned to him in March of that year. He held the chair for only a few months, as a result of his sudden death. A new *Oposicion* was declared and three candidates applied: Carlos de Siguenza y Góngora, Juan de Saucedo, and Joseph Salmeron de Castro. Before the public appearance, Salmeron de Castro registered a protest, declaring that he was the only qualified candidate, as neither of the other two were bachelors as required by the constitution. The contention was not sustained, Siguenza y Góngora arguing that Muñoz had not been a bachelor in astrology and mathematics either.

On July 17, 1672, Juan de Zaucedo was assigned a subject and he read for an hour on *de ambitu terrae*. Don Carlos de Siguenza y Góngora read on the nineteenth on *de ortu et occasu Signorum*, and Salmeron de Castro read on the twenty-first on *de Zodiaco Circulo*. When the votes were counted it turned out that Salmeron de Castro had 14 votes; Don Carlos de Siguenza y Góngora, 74; and Juan de Saucedo, 7. The chair was given to Siguenza y Góngora, who had received 60 votes more than his nearest competitor.

The salary attached to the chair was not munificent, even if we consider the much greater purchasing power of money at that time. According to the minutes of the university, Siguenza was assigned 100 pesos a year.

The choice was a happy one. Don Carlos de Siguenza y Góngora held this chair for eighteen years. During this time he acquired the reputation of being one of the great scientists of his day and certainly the equal of any in Europe. Strange as it may seem, this professor of mathematics and astrology, later honored with the high-sounding title of Royal Cosmographer, was one of the first public men of Mexico to become deeply interested in far-away Texas. It was to him that Father Massanet wrote his famous long letter giving a detailed account of the first *entrada* into Texas in 1689 with Alonso de Leon, in which are found numerous details of the Indians and the country as it then was. It is claimed that Don Carlos wrote the first history of Texas, but unfortunately the manuscript has been lost for these many years. If it is ever found, it will, no doubt, be very interesting reading.

Don Carlos became well known as the result of his heated controversy with no other personage than Father Kino of California fame. This scholar, but recently from an Austrian university, had a low opinion of the knowledge of the Mexican savant. They entered into a polemic that has become famous, and curious to say, though Kino was given the palm of victory by his contemporaries, modern scientists have shown that fundamentally the despised Mexican savant was correct. He was the first to disprove seriously and with well founded arguments the long-standing superstition that comets brought in their wake pestilence, war, and death.

NUMERICAL ASPECTS OF CERTAIN GEOMETRIC FORMULAS

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INTRODUCTION

Mr. O. S. Hollabaugh of the Vickery High School states that, in connection with solid geometry problems concerning the diagonal of a room, he found that the diagonal is an integer if the dimensions of the room are two consecutive integers and their product. For example, $2^2 + 3^2 + 6^2 = 7^2$. His proof of this interesting result appeared in the November number of this BULLETIN. That there are solutions in integers of the equation $a^2 + b^2 + c^2 = d^2$ other than those given by Mr. Hollabaugh's method is shown by the fact that $1^2 + 4^2 + 8^2 = 9^2$, and it is a problem in the theory of numbers to find a method for obtaining all sets of integers satisfying the equation.

Theorems of geometry give the three equations:

I. $a^2 + b^2 = c^2$, c the hypotenuse of a right triangle and a and b the other two sides.

II. $a^2 + b^2 + c^2 = d^2$, d the diagonal of a rectangular parallelopiped with dimensions a, b, and c.

III. $K = \sqrt{s(s-a)(s-b)(s-c)}$, K the area of a triangle with sides a, b, and c, and $2s = a + b + c$.

Teachers may be interested in methods of finding solutions of these equations in which all quantities are integers. In this paper theorems are stated, and in two cases proved, by which solutions of these equations can be found, and tables showing several solutions for each equation are given. The proofs involve only algebra and some results of the elementary theory of numbers which are so simple that, I believe, a thoughtful person can provide demonstrations of them for himself in case they are not obvious. Hence no extensive mathematical knowledge is necessary for the understanding of this discussion.

After the consideration of the three equations that have been mentioned, a method is indicated for proving that it is impossible to construct a regular pyramid on a square base with all edges rational and with both volume and lateral area rational.

The word "number" means integer wherever used in this paper. "Greatest common factor" is abbreviated to "g.c.f."

I. $a^2 + b^2 = c^2$

A right triangle with all sides integers is called a Pythagorean triangle. If the g.c.f. of the sides is 1, the triangle is said to be "primitive." Since all other Pythagorean triangles may be obtained from the primitive triangles (as $9^2 + 12^2 = 15^2$ from $3^2 + 4^2 = 5^2$), a method for getting all "primitive" solutions of (I) will be sufficient. A proof will be given of the theorem:

Theorem: If x, y, and z are the legs and hypotenuse respectively of a primitive Pythagorean triangle, there is a pair of numbers m and n, one odd, one even, with g.c.f. $= 1$, such that $x = 2mn$, $y = m^2 - n^2$, and $z = m^2 + n^2$; and conversely, every such pair of numbers m and n yields a primitive Pythagorean triangle with sides represented by $2mn$, $m^2 - n^2$, and $m^2 + n^2$.

Proof: Consider (1) $x^2 + y^2 = z^2$, x, y, and z sides of a primitive Pythagorean triangle.

Since the triangle is primitive, the g.c.f. of the three numbers is 1. To prove that the g.c.f. of any two of them is 1, suppose two of these numbers have a common factor $d > 1$. Then d^2 is a factor of the sum and difference of their squares, one of which is the square of the third number. Hence d is a factor of the third number also, and the triangle is not primitive. This contradiction shows that no two of the numbers have a common factor > 1. Hence x and y cannot both be even. To show that they cannot both be odd, we must note that any odd number can be written as $2k + 1$, and its square, $4k^2 + 4k + 1$, can be written $4j + 1$. Then the sum of the squares of two odd numbers is divisible by 2 but not by 4, and cannot be a square, since the square of an even number is divisible by

4. Since $x^2 + y^2$ is a square, x and y cannot both be odd. Hence one is odd and one is even. Let x represent the even number. Then y and z are odd.

From (1) $x^2 + y^2 = z^2$,

(2) $x^2 = z^2 - y^2 = (z + y)(z - y)$.

A common factor of $z + y$ and $z - y$ is a factor of both $2z$ and $2y$, their sum and difference respectively. Since the g.c.f. of y and z is 1, the g.c.f. of $z + y$ and $z - y$ is 2. Their product is a square, so it can be seen that each is twice a square. Write (3) $z + y = 2m^2$ and $z - y = 2n^2$. The preceding discussion shows that the g.c.f. of m and n is 1.

From (2), $x = 2mn$.

From (3), $y = m^2 - n^2$ and $z = m^2 + n^2$.

If m and n were both odd or both even, z would be even, which is impossible. Therefore one of m and n is odd and one even.

This proves that any numbers x, y, and z which are the sides of a primitive Pythagorean triangle may be written

$$x = 2mn,\ y = m^2 - n^2,\ z = m^2 + n^2,$$

m and n having g.c.f. $= 1$, and one odd, one even. This is the first part of the theorem concerning these triangles which has been stated.

As for the converse, from elementary algebra, $(2mn)^2 + (m^2 - n^2)^2 = (m^2 + n^2)^2$. It remains to be shown that if m and n have g.c.f. $= 1$, and one is odd, one even, the resulting Pythagorean triangle is primitive—that is, $2mn$, $m^2 - n^2$, and $m^2 + n^2$ have g.c.f. $= 1$. Suppose the last two numbers have a common factor $d > 1$. Then d is a factor of both their sum and their difference, and hence a factor of both $2m^2$ and $2n^2$. Since $m^2 + n^2$ is odd (the sum of the squares of an odd and an even number is odd), d cannot be 2. Therefore d is a factor of both m^2 and n^2, hence of both m and n. This contradicts the requirement that the g.c.f. of m and n be 1. The contradiction shows that the three numbers $2mn$, $m^2 - n^2$, and $m^2 + n^2$ have g.c.f. $= 1$, and form a primitive Pythagorean triangle. This completes the proof.

This theorem was known to Diophantos of Alexandria, who lived in the third century A. D. The proof given here is standard in the theory of numbers. I have followed the form found in Carmichael's *Theory of Numbers.*

In the following table several Pythagorean triangles are shown; m and n are the parameters, a, b, and c the sides of the triangles, and $a^2 + b^2 = c^2$. One example in large numbers is given:

m	n	a	b	c
2	1	4	3	5
3	2	12	5	13
4	1	8	15	17
4	3	24	7	25
5	2	20	21	29
5	4	40	9	41
6	1	12	35	37
6	5	60	11	61
7	2	28	45	53
7	4	56	33	65
7	6	84	13	85
30	23	1,380	371	1,429

II. $a^2 + b^2 + c^2 = d^2$.

Mr. Hollabaugh has shown that if a, b, and c are two consecutive numbers and their product, the sum of their squares is a square—a result, it is interesting to note, which was tacitly assumed by Diophantos and is supposed to have been one of his theorems which are now lost. That there are other solutions has been shown by an example, and a method will be given for finding all "primitive" solutions of this equation, a primitive solution being a solution such that the g.c.f. of a, b, c, and d is 1. Other solutions can be easily obtained from the primitive solutions.

If the g.c.f. of three of a, b, c, and d is $k > 1$, then k^2 is a factor of the squares of these three numbers, and hence of every sum and difference of their squares different from zero. Since the square of the fourth number is such a combination of their squares, k^2 is a factor of the square of the fourth number and hence k is a factor of the fourth number. Thus we see that in a primitive solution no three of the numbers have a g.c.f. > 1.

The following theorem will give a method of calculating all primitive solutions of the equation.

Theorem: For every pair of numbers a and b, one odd and one even, there are numbers c and d such that $a^2 + b^2 = d^2 - c^2$, where a, b, c, and d have the g.c.f. 1, and hence a primitive solution of

$$\text{(II)} \quad a^2 + b^2 + c^2 = d^2.$$

Conversely, all primitive solutions of (II) can be written $a^2 + b^2 = d^2 - c^2$, one of a and b odd, one even.

Proof: Let $a^2 + b^2 = N$, where one of a and b is odd and one even; then N is odd. Write $N = pq$, where $p > q$ (a matter of notation), and the g.c.f. of a, b, p, and q is 1. One such pair always exists, since $p = N$ and $q = 1$ is admissible. p and q are both odd, since N is odd; hence their sum and difference are even. Let $2d = p + q$, $2c = p - q$. Then $d + c = p$, $d - c = q$, and $N = d^2 - c^2$. Hence $a^2 + b^2 + c^2 = d^2$.

If c and d have a g.c.f. $k > 1$, then p and q, their sum and difference, have k as a common factor. Since the g.c.f. of a, b, p, and q is 1, the g.c.f. of a, b, c, and d is 1, and the solution of (II) which we have obtained is primitive.

For each distinct pair of factors of N satisfying the conditions above we find a primitive solution of (II). It now remains only to show that every primitive solution can be obtained by this scheme, as is stated in the second part of the theorem. To prove this it will be shown that in every primitive solution of $a^2 + b^2 + c^2 = d^2$, d is odd, and exactly one of a, b, and c is odd. Then we shall obtain, by transposing an even term, the sum of two squares, one odd, one even, equal to the difference of two squares. This solution will be found by the method given in the preceding paragraphs, and thus it will be shown that all primitive solutions are obtained by the scheme.

To show that exactly one of a, b, and c is odd, note that the square of an even number is divisible by 4, but the square of an odd number has the form $4k + 1$, k an integer. (That is, the remainder 1 is left if an odd square is divided by 4; $(2m + 1)^2 = 4m^2 + 4m + 1 = 4k + 1$.) a, b, and c cannot all be even, for the g.c.f. of any three of a, b, c, and d is 1. If all three were odd, then $a^2 + b^2 + c^2 = 4k + 3$, and this cannot be d^2. If one were even and two odd, $a^2 + b^2 + c^2 = 4k + 2$, and this cannot be d^2, since it is divisible by 2 but not by 4. The only remaining possibility is that one is odd and the other two even. Then d^2 is odd, and d is odd.

This completes the proof.

Some sets of integers satisfying $a^2 + b^2 + c^2 = d^2$ are shown in the table below. An example of calculation is displayed. Unless two of a, b, and c are equal (as in $1^2 + 2^2 + 2^2 = 3^2$), the set will be obtained twice as all admissible pairs of a and b are taken in some order. (*) marks duplicates.

Sample calculation:

$$2^2 + 9^2 = 85 = 85 \,.\, 1 = (43 + 42)(43 - 42) = 43^2 - 42^2$$
$$= 17 \,.\, 5 = (11 + 6)(11 - 6) = 11^2 - 6^2.$$

$$\text{Hence } 2^2 + 9^2 + 42^2 = 43^2$$
$$\text{and } 2^2 + 6^2 + 9^2 = 11^2$$

a	b	c	d
1	2	2	3
2	3	6	7
1	4	8	9
3	4	12	13
2	5	14	15
4	5	20	21
1	6	18	19
3	6	22	23
3	6	2	7*
5	6	30	31
2	7	26	27
4	7	32	33
4	7	4	9*

III. $K = \sqrt{s(s - a)(s - b)(s - c)}$

In the *American Mathematical Monthly* for January, 1929, Dr. Wm. Fitch Cheney, Jr. discusses triangles with rational area K and integral sides a, b, c, the g.c.f. of the sides 1, under the name of Heronian triangles. He proves a number of theorems concerning them, and although the proofs are too long to repeat here, the following theorems may be of interest. (The numbering of the article is retained.)

Th. 14: The semi-perimeter of any Heronian triangle is an integer.

Th. 15: Any Heronian triangle has just one even side.

Th. 16: The area of any Heronian triangle is an integer.

Th. 18: The area of any Heronian triangle is even.

Th. 19: The area of any Heronian triangle is a multiple of 3.

Th. 20: In any Heronian triangle, if the area is not a multiple of 5, one of the sides is.

Heronian triangles can be formed by the juxtaposition of two Pythagorean triangles. (For example, right triangles with sides 5, 12, and 13, and 9, 12, and 15, may be placed to form the Heronian triangle with sides 13, 14, and 15.) However, some Heronian triangles cannot be formed by this method. Mr. Cheney gives a method of finding all Heronian triangles by considering the tangents of the half-angles of the triangles. The following list, arranged according to the order in which they are obtained by Mr. Cheney's method, is taken from his article. Two right triangles are omitted.

a	*b*	*c*	*K*
5	5	6	12
10	17	21	84
5	5	8	12
25	52	63	630
40	51	77	928
13	20	21	126
15	37	44	264
25	39	56	420
17	17	30	120
26	51	55	660
25	29	36	360
35	100	117	1,638
20	37	51	306
85	104	171	3,420
87	100	143	4,290
68	75	77	2,310
4	13	15	24
7	15	20	42
51	74	115	1,380
13	37	40	240
13	13	24	60
58	85	117	2,340
25	51	52	784
35	53	66	924
17	25	28	210
65	87	88	2,640

ANOTHER PROBLEM

Problems in geometry give rise to equations other than those which have been discussed. For example, a teacher who is assigning problems in volumes and areas of pyramids may wish to give the class the lengths of the edges of a regular pyramid on a square base and to ask the pupils to calculate the lateral area and volume. If the lengths of the edges are rational, is it possible to have both lateral area and volume rational? The answer is that this is impossible. An outline of a proof follows. (The omitted part is given in full, although not in connection with this problem, both in Carmichael's *Theory of Numbers*, pp. 86–89, and in *Diophantine Analysis* by the same author, pp. 14–16.)

If we have a regular pyramid on a square base with all edges rational, we may obtain a regular pyramid on a square base with all edges integers and the sides of the base even by multiplying each edge by twice the lowest common denominator of the edges. Since the lateral area is $\frac{1}{2}ps$, p the perimeter of the base and s the slant height, s must be rational if the lateral area is rational. Likewise, the altitude, h, of the pyramid must be rational if the volume is rational. If a pyramid of the sort that we are considering has both s and h rational, we can obtain one with both s and h integers by multiplying each edge by the lowest common denominator of s and h. Let a lateral edge be of length j and a side of the base of length $2a$. (The first multiplication made the sides of the base even.) Then we have these equations:

$$(1)\quad a^2 + h^2 = s^2, \qquad (2)\quad a^2 + s^2 = j^2,$$

a, h, s, and j all integers.

That the existence of such a set of integers is impossible can be shown by the famous method of "infinite descent," which is due to Fermat. Suppose that there is a set of integers satisfying these equations. By certain substitutions and some manipulation, a second set of integers is found satisfying the equations, with each number of the second set less than the corresponding number of the first set. By a repetition of the process, a third such set can be found, with each number less than the corresponding number of the second set. This can be continued *ad infinitum.* Thus corresponding to a of the first set, we have a_2 of the second set, and $a_2 < a$; a_3 of the third set, $a_3 < a_2$; and so on indefinitely. But this is impossible, since there is only a finite number of distinct integers less than a. Thus we have a contradiction proceeding from the assumption that a set of integers exists satisfying the equations, and hence the existence of such a set is impossible.

Incidentally, it may be of interest to note that the impossibility of such a set of integers, together with the theorem that has been stated concerning the sides of a primitive Pythagorean triangle, may be used to prove that the area of a Pythagorean triangle is never a square number.

THE AIM OF THE GEOMETRY COURSE

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It is generally agreed that the main justification for requiring all high-school students to study mathematics is its disciplinary value. To be sure, mathematics has a great deal of practical value; it lies at the basis of our whole machine-age civilization, which would be impossible without it. The fundamental importance of mathematics to modern civilization does not imply, however, that knowledge of mathematics is equally important to each individual participant in that civilization, for a man may turn the dial of his radio and listen to a jazz orchestra playing a thousand miles away without any comprehension of the long series of physical and mathematical discoveries which made such a marvelous achievement possible. The amount of use which the average business or professional man makes of algebra and geometry does not warrant compelling all students to learn these subjects. It is only because it develops or is supposed to develop the reasoning powers that mathematics is given an important place in the curriculum. The ability to reason clearly and accurately is of vastly greater value in later life than any amount of mere information.

For the accomplishment of this purpose geometry is better adapted than algebra. In algebra the formal manipulation of symbols necessarily plays a considerable part, but geometry can—and should—be made almost wholly a course in pure reasoning. Unfortunately the true aim of the study is often completely lost sight of; many a student endowed with a good memory has passed the geometry course with flying colors who could not work the simplest "original." More than once have I heard from students unable to pass their freshman college mathematics the statement that "Math. was my easiest subject in high school."

In order that the geometry course may fulfil its *raison d'être,* the main emphasis should be placed on the original exercises, for it is in solving these that the student has the best opportunity to use and develop his ability to reason. It is only too apparent from the results usually obtained in our schools that the memorizing of the proofs given in the textbook, models of perfect reasoning though they may be, is lamentably inadequate to teach the student to think for himself. As in most other activities, both physical and mental, one can learn to reason only by reasoning, and proficiency comes from practice.

Original exercises have the great advantage that they can be graded in order of difficulty. In the textbook propositions logical considerations sometimes require that a long or difficult proof be placed near the beginning, but the exercises can be graduated from those which involve nothing more than the mechanical application of the preceding theorem, and which can be worked by the dullest pupil, to those which test the utmost ingenuity of the best scholar in the class (or even of the teacher). Sometimes a difficult theorem can be made much easier by means of a series of exercises which lead up to it by gradual steps.[1]

The first month or so of the geometry course should be devoted to introductory matter (unless the class has previously had a course in "intuitive geometry"), designed to familiarize the students with the fundamental notions of geometry, the use of the instruments, etc. With the beginning of formal demonstrations, however, originals should be assigned, and the teacher should keep constantly in mind the fact that his main object in the course is to develop the pupils' ability to work original exercises of gradually increasing difficulty and complexity.

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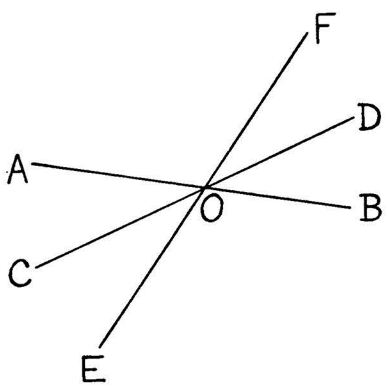
[1]For examples of how this can be done, see Schultze: *The Teaching of Mathematics in Secondary Schools,* pp. 101–103.

If the first theorem taken up is that of the equality of vertical angles, simple exercises can be constructed from the figure formed by three lines meeting at a point; e. g., prove that the angle *AOD* is the sum of angles *AOF* and *COE;* that the sum of angles *COA*, *FOD*, and *BOE* is a straight angle; that if *CD* bisects the angle *AOE*, it bisects also its vertical angle *FOB*.



Such exercises furnish a good introduction to the working of originals.

As soon as the theorems on the congruence of triangles have been proved, a very wide variety of exercises is available, involving the proving of line-segments or angles equal by the aid of congruent triangles. These are thoroughly typical of originals in general, since this is the most commonly used method of proving segments and angles equal; at the same time they can be made simple enough to come within the ability of every student. At first exercises can be given in which the only triangles in the figure are those which are to be proved congruent; later, the figures may involve several triangles, from which the student has to pick out the pair he needs; and finally, exercises may be given in which one or more lines must be drawn in order to obtain the congruent triangles. A thorough drill on such problems at this time will go far toward giving the student a comprehension of the true nature of geometry and ability to carry out the necessary logical steps.

Training of this sort if well done will also tend to arouse and maintain the students' interest in geometry. Boys and girls enjoy puzzles, and are willing to spend much time and effort on them if a reasonable degree of success can be attained, as was shown by the craze for cross-word puzzles a few years ago. This mental attitude should be capitalized in the geometry course.

It is much easier today to teach geometry as it should be taught than it was a few years ago, for the movement for the reform of mathematical teaching which began about twenty-five years ago has led the writers of textbooks to lay increasing emphasis on the exercises, and they fill a large and prominent place in the best of the recent texts. Even the details of the proofs of the regular propositions are often left for the student to supply. Even so, the teacher would do well to have one or two books at hand, in addition to the text which the class is using, as sources of extra problems, especially if much blackboard work is done. It is likewise advantageous for the teacher to be able to invent problems extemporaneously in class. Suggestions along this line, as well as good lists of supplementary exercises, will be found in the chapters on geometry in A. Schultze's *The Teaching of Mathematics in Secondary Schools* (The Macmillan Company, New York). Teachers who wish to develop their own ability to work difficult originals will find material in E. S. Loomis's *Original Investigation, or How to Attack an Exercise in Geometry* (Ginn and Company, Boston), and J. Petersen's *Methods and Theories for the Solution of Problems of Geometrical Constructions* (G. E. Stechert and Company, New York).

Recently I came upon an appreciation of geometry from a wholly unexpected source. Mahatma M. K. Gandhi, the Hindu saint and statesman, and one of the greatest religious geniuses of all time, has the following to say in his autobiography.[2]

> English became the medium of instruction in most subjects from the fourth standard onward, and at first I found myself completely at sea. Geometry was a new subject in which I was not particularly strong, and the English medium of instruction made it still more difficult for me. The teacher taught the subject well, but I could not follow him. . . . When, however, with much effort I reached the thirteenth proposition of Euclid, the utter simplicity of it all was suddenly

[2]From *Mahatma Gandhi—His Own Story*. Edited by C. F. Andrews. By permission of The Macmillan Company, publishers, New York. P. 51.

revealed to me. A subject which only required the simple use of one's reasoning powers could not be difficult. Ever since that time geometry has been both easy and interesting for me.

Sanskrit however proved a harder task. In geometry, there was nothing to memorise; whereas in Sanskrit everything had to be learnt by heart.

THE POISSON STATISTICAL LAW

EDWARD L. DODD
The University of Texas

In the earlier treatments of probability and statistics, the so-called normal law of distribution plays a central and an almost exclusive rôle. The equation for this distribution is variously written:

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}; \quad y = \frac{1}{s\sqrt{2\pi}} e^{-x^2/2s^2}$$

with h as the measure of precision and s the standard deviation. This yields a symmetrical bell-shaped or mound-shaped curve, with its highest point above the origin. On both sides the curve slopes off like the sides of a hill.

As statistical science advanced, it became evident that this simple curve could not adequately describe various distributions arising in practice. Generalizations were made. Karl Pearson, of England, devised a system of curves or "types" of great flexibility. Other writers made effective use of the "normal" function together with its derivatives. It is my purpose, however, to describe briefly a distribution law, distinct from these, a law associated with the name of Poisson. Great prominence is given the Poisson Law by T. C. Fry, member of the technical staff of the Bell Telephone Laboratories, in his recent book: *Probability and Its Engineering Uses,* D. Van Nostrand and Co.

Like the Normal Law, the Poisson Law is an approximation for the point binomial; but the Poisson Law postulates also that the probability for occurrence in an individual case is relatively small.

Problems satisfying this assumption arise frequently. For life insurance, the probability that a man of age 40 will die within a year is commonly taken as 0.01; for the younger ages it is much smaller, and even at age 63, it is only 0.03. In connection with telephone traffic—with the possibilities of congestion—the probability that a given subscriber will be using his phone in a specified minute may be rather small.

Many college algebras give the reader a start in the theory of probability. If p is the probability that an event will occur and q is the probability that it will not occur, then $q + p = 1$. If now, we raise the binomial $(q + p)$ to the nth power,

$$(q + p)^n = q^n + npq^{n-1} + \dots + np^{n-1}q + p^n,$$

the terms in succession give the probabilities for 0, 1, ..., $(n - 1)$, n occurrences of an event in n trials—or for this number of instances of occurrence among n cases of like nature. Thus, the successive terms of

$$(.99 + .01)^3 = (.99)^3 + 3(.99)^2(.01) + 3(.99)(.01)^2 + (.01)^3$$

give respectively the probabilities that exactly 0, 1, 2, and 3 men of age 40 will die within a year.

Now the Poisson approximations for these terms are found by first taking $np = 3(.01) = .03$; and then writing

$$e^{-.03} + e^{-.03}(.03) + e^{-.03}(.03)^2/2 + e^{-.03}(.03)^3/6.$$

PROBABILITY OF r DEATHS

r	By Binomial	By Poisson
0	.9703	.9704+
1	.0294	.0291+
2	.0003	.0004+
3	.0000	.0000
	1.0000	1.0000

In the general case where there are n trials of an event with individual probability p, the so-called "expected" number a of occurrences is

$$a = np$$

The Poisson Law gives as the successive probabilities for 0, 1, 2, ... occurrences, the terms of the development:

$$1 = e^{-a} \cdot e^{a} = e^{-a}\left(1 + \frac{a}{1} + \frac{a^2}{2!} + \frac{a^3}{3!} + \ldots.\right)$$
$$= e^{-a} + e^{-a}\frac{a}{1} + \ldots. + e^{-a}\frac{a^r}{r!} + \ldots.$$

As another illustration of the application of the Poisson Law, I will reproduce in simplified form a table given by T. C. Fry on page 295 of his aforementioned *Probability.* The "Expected Frequency" will be rounded off to the nearest integer. The first four frequencies, constituting only about 1 per cent of the total frequency of 3754 calls will be lumped together. Likewise, the last four frequencies.

The deviation d is the difference between the observed frequency and the expected frequency as given by the Poisson Law; where 10.44, the average number of busy senders, was taken as a fair estimate of the expected number, corresponding to a in the Poisson formula. Moreover, letting E be the expected frequency for a specified class, the divergence D of that class is $D = d^2/E$.

NUMBER OF BUSY SENDERS IN A TELEPHONE EXCHANGE

(1) Number Busy	(2) Observed Frequency	(3) Expected Frequency	(4) Deviation d −	(4) Deviation d +	(5) Divergence D
0–3	43	28	----	15	8.0
4	57	54	----	3	0.2
5	111	113	2	----	0.0
6	197	197	----	0	0.0
7	278	294	16	----	0.9
8	378	384	6	----	0.1
9	418	446	28	----	1.8
10	461	465	4	----	0.0
11	433	442	9	----	0.2
12	413	384	----	29	2.2
13	358	309	----	49	7.8
14	219	230	11	----	0.5
15	145	160	15	----	1.4
16	109	104	----	5	0.2
17	57	64	7	----	0.8
18	43	37	----	6	1.0
19–22	34	41	7	----	1.2
	3754	3752	−105	+107	26.3

The expected frequencies given in Column 3 are a fair approximation for the observed frequencies in Column 2.

This follows from the fact that the sum 26.3 of the items in the Divergence Column 5 does not greatly exceed 17, the number of classes involved.

As large a divergence as 26.3 might happen by pure chance about once in twenty times. The first item of 8.0 in the Divergence Column is somewhat large; and there may be some special cause for an excess in the corresponding frequencies. Apart from this, the fit is a decidedly good one.

Another illustration I shall take from Fry's *Probability,* which was indeed not given to illustrate the Poisson Law, but which I found did rather closely follow this law. It was called "an elementary problem in sampling." On page 129, we find: "A factory produces a certain type of screw as a standard product. The screws are collected at the machine in boxes of 1,200 each. Long experience has shown that the proportion of these boxes which contain various percentages of bad screws is substantially as follows:

Per Cent of Bad Screws in the Box	Proportion of Boxes Observed to Contain This Percentage of Bad Screws
0	0.78
1	0.17
2	0.034
3	0.009
4	0.005
5	0.002
6	0.000

Two per cent badness has been adopted as a manufacturing standard; that is, any box which contains 2 per cent or less of bad screws is regarded as satisfactory, the aim of the inspection process being to reject those which are poorer. The normal inspection consists in the examination of 50 screws out of each box. A particular box, produced at a time when there was no special reason to suspect that the machines were not operating properly, showed 6 bad screws under normal inspection. What is the probability that the manufacturing standard had not been maintained in the production of this box?"

This probability turns out to be 0.926; and thus it would seem worth while to examine the machinery to locate the trouble. But the point I wish to emphasize here is that the probability that a specified screw should be bad is comparatively small—comparable with the probabilities for death that I mentioned in the first illustration—and thus there is *a priori* reason for supposing that a Poisson distribution could be found giving a good fit.

In the concluding chapter, Fry deals with the kinetic theory of gases, and arrives at the Maxwell distribution of velocities in the usual "normal form." However, here also he finds problems which lead to the Poisson Law, in particular, relating to density fluctuations. Dealing with a cubical element of volume 0.01 cm. on a side, within a gas at room temperature and atmospheric pressure, he notes that there is very little chance of any appreciable change of density. "If, however, we were to consider a cube the dimensions of which were comparable to a wave-length of light we would find that the density fluctuations were appreciable, and if we were to deal with a fluid in which colloidal particles were suspended there might be considerable fluctuations in even larger elements. Such fluctuations are believed to be the cause of the optical phenomenon known as opalescence."

Among the tables which appear in the appendix of Fry's *Probability* is one giving individual probabilities by the Poisson formula, and another giving the sums of such of these probabilities as are required frequently in engineering problems.

It is beyond the scope of this paper to go into details of proof. Fry shows that many paths lead to the Poisson Law. One of the simplest of these makes use of the representative term of the binominal $(q+p)^n$ heretofore discussed, viz.

$$\binom{n}{r} p^r q^{n-r}$$

where the parenthesis is the binomial coefficient giving the number of ways of selecting r objects from n objects. With p small, the only significant terms arise when r is small relative to n; and thus,

$$\binom{n}{r} = \frac{n^r}{r!}$$

approximately. Moreover, with small p

$$q^n = (1 - p)^n = e^{-np}$$

approximately; and q^{-r} is close to unity.

It is not the purpose of this paper to compare the relative importance of various systems of frequency curves, but merely to call attention to one distribution curve which has many important practical uses—the Poisson Law or "Poisson Exponential," as it is frequently called.

RELATION BETWEEN ACHIEVEMENT AND ABILITY OF PUPILS IN FIRST SEMESTER ALGEBRA IN PORT ARTHUR HIGH SCHOOL

F. D. AUSTIN, DIRECTOR OF MATHEMATICS

LOIS PARKER, INSTRUCTOR

Port Arthur High School

It has been our belief for the past year that the teaching in all subjects has been primarily for the unprivileged child, while the talented ones have been greatly neglected. Most colleges offer courses on "How to Teach the Unprivileged Child," but few give instructions on how to teach the gifted ones and are therefore greatly responsible for this unbalanced teaching in the public schools. We see no reasons why the better pupils should not demand just as much attention as the slower group. That this situation is real has been proved beyond a doubt to our own satisfaction in both algebra and geometry. This article, in which we attempt to tell how we determined whether or not the students were working up to their mental capacity, pertains only to first semester algebra pupils.

At the end of the first semester of this school year, a two-hour examination was given to all first-semester pupils taking first year algebra (151 pupils). In making up this test we kept in mind the four qualifications of every good test—comprehensiveness, validity, objectiveness, and reliability. As to comprehensiveness and validity, this test covered every fundamental concept in the course of study which has been approved by the Committee on Reorganization of Mathematics in Secondary Schools and by the best present-day practice as determined by leading educators, psychologists, and mathematics teachers.

The examination was purely objective to grade. After the tests were scored we obtained the following information in determining the reliability of the tests:

Reliability	.90
Standard Deviation	37.3
Probable error (score)	7.5
Highest possible score for test	174
Highest mark earned	168
Lowest mark earned	18

As no pupil made a perfect score and no pupil made zero, it is clearly seen that both the best pupils' and the poorest pupils' algebraic achievement was measured.

With this as a background the correlation was run between the scores made on this test and the Intelligence Quotients of the pupils as determined by the Otis and Lee Intelligence Test. The coefficient of correlation as determined by the Pearsonian Method was found to be .56 with a probable error of .04. This is good correlation for a group, but did not tell if the individual pupil was working up to his mental capacity. To determine this we ranked the pupils according to both Intelligence Quotients and score made on the test.

The range of the Intelligence Quotients of these 151 pupils was from 64 to 132. The pupil with the highest Intelligence Quotient was ranked first; the next highest second, and so on to the lowest Intelligence Quotient, which ranked 151. These pupils were also ranked according to the score made on the test. The highest score, which was 168, was ranked first; the next highest second, and therefore the lowest score, which was 18, ranked 151.

If these correlations were perfect the pupil having the highest Intelligence Quotient would have made the highest score on the test, and so on. In other words, each pupil's rank in Intelligence Quotient and test score should have corresponded. This we found not at all the case, as is shown below in the results of ten of the 151 pupils.

Name of Pupil	Intelligence Quotient	Score on Test	Rank of Intelligence Quotient	Rank of Test
A	132	144	1	22.5
B	131	114	2	56
C	127	160	3	8
D	125	163	4.5	2
E	125	160	4.5	8
F	124	138	6	29.5
G	123	145	8	19.5
H	123	130	8	35
I	123	138	8	29.5
J	122	160	10.5	8

From this one can see that pupil B is working 54 points below his mental capacity and should therefore have special attention. The trouble may be one of many things, as health, reading ability, laziness, home environment, or the teacher's failure to appeal. This boy is not a failure in the usual sense of the word, but is a bad failure in not doing what he is capable of, if all other things are equal. A chart of this kind for the 151 pupils certainly did enable us to give some individual instruction to the pupils who needed it, rather than to those who made failing grades. The latter may be doing all they are capable of, and should be consulted about taking the course in Business or Shop Mathematics which is also offered in this school.

We then divided the pupils into two groups—those with Intelligence Quotients of 95 and above, and those with Intelligence Quotients below 95. An Intelligence Quotient of 95 was considered as denoting normal ability. From a study of these two groups we found the following results:

STUDENTS WITH INTELLIGENCE QUOTIENTS OF 95 AND ABOVE

	Number	Per Cent
Students who rank higher in achievement than I.Q.	30	28
Students who rank lower in achievement than I.Q.	73	68
Students who rank nearly the same	4	4
Total	107	100

STUDENTS WITH INTELLIGENCE QUOTIENTS BELOW 95

	Number	Per Cent
Students who rank higher in achievement than I.Q.	34	77
Students who rank lower in achievement than I.Q.	7	16
Students who rank nearly the same	3	7
Total	44	100

Therefore, since 68 per cent of the better group of pupils and only 16 per cent of the lower group were working below capacity, it is clearly seen that in Port Arthur more attention should be given the better class of pupils without lowering the quality of work of the slower group.

Results similar to these were also obtained from the geometry tests.

THE DEMAND AND SUPPLY OF HIGH-SCHOOL TEACHERS OF MATHEMATICS

MIRIAM DOZIER

Secretary, Teachers Appointment Committee
The University of Texas

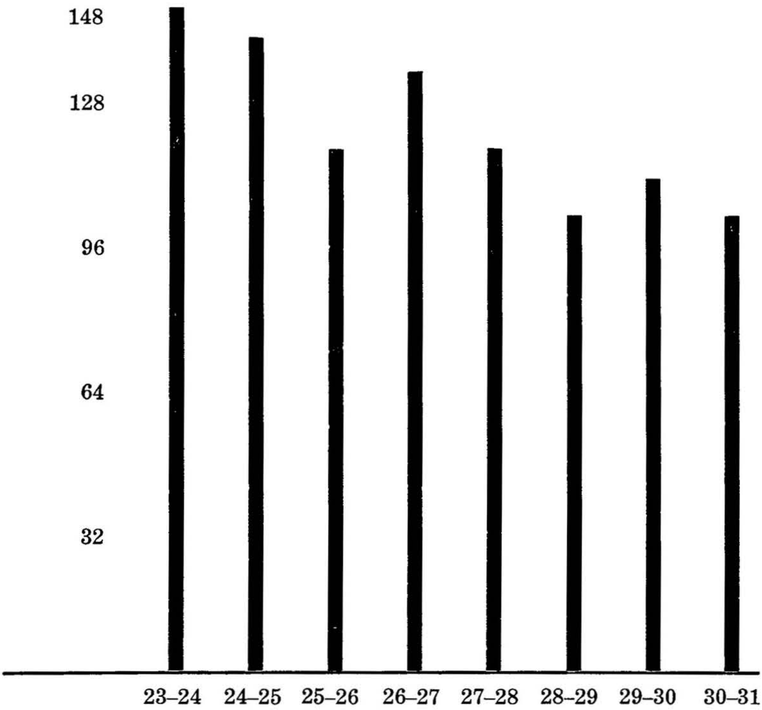
It has occurred to the writer that it might be of interest to the teachers of mathematics in the high schools of the state to compare statistics showing the demand for teachers of this subject with those showing the number of applicants who have registered for positions to teach it. These figures cannot, of course, be construed to represent the total demand and supply of teachers of mathematics in Texas; yet, these figures can be studied comparatively so as to show the trend of teacher-supply in this particular field of teaching.

With this in view, there are presented herewith Graphs I and II on the following pages. Figures are available for a much longer period than the eight years included here as regards the demand for teachers; unfortunately, they are not available for the number of teachers seeking positions in mathematics through our service. If the attention be centered on Graph I, there is seen to be an almost steady decrease from year to year in the number of vacancies in this subject. Exceptions are to be seen for the years 1926–27 and 1929–30. This doubtless means that teachers are now more aggressive in seeking positions than formerly, and this makes unnecessary their active pursuit by school officials. This aggressiveness is necessitated, it seems, by the everincreasing number of teachers who are available for positions.

If the attention now be turned to Graph II, it is to be noted that the number of teachers seeking positions in mathematics has remained fairly constant through the past eight years. This appears contradictory to the statement made in the last paragraph relative to the ever-increasing number of teachers. The real point here is that the total aggregate number of teachers, or of those wanting to teach, through the past eight years has grown larger, but the number applying for mathematics has remained practically unchanged.

GRAPH I

Showing Number of Calls for High-school Teachers of Mathematics 1923–24 to 1930–31

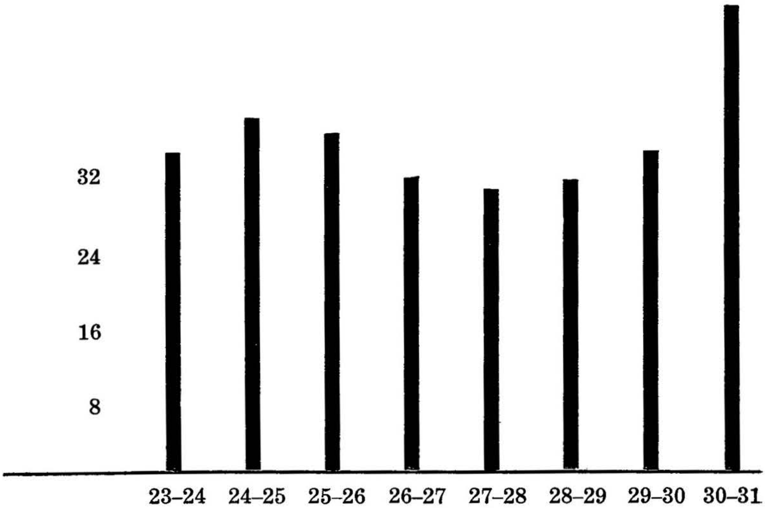


1923–24	148
1924–25	143
1925–26	116
1926–27	136
1927–28	117
1928–29	100
1929–30	108
1930–31	98

Other statistics compiled from the records of the Committee, but not presented here, show that the number of teachers applying for work in science through the same years has also remained fairly constant, whereas the number of those applying for other subjects of the curriculum, such as English, history, home economics, commercial subjects, and Spanish has increased yearly.

GRAPH II

Showing Number of High-school Teachers Available for Mathematics 1923–24 to 1930–31



1923–24	34
1924–25	38
1925–26	36
1926–27	31
1927–28	30
1928–29	31
1929–30	35
1930–31	50

A comparison of the figures on the basis of which the two graphs were constructed shows a wide disparity between the demand and the supply available. This is true for every year of the period under consideration. One might be tempted, therefore, to conclude immediately that anyone registering with the Committee would have no difficulty in getting placed. Before an affirmative assent may be given to this conjecture, it is well to call the attention of interested readers to several considerations which must be faced in the matter of placing teachers. In the first place, the Committee does not have the right of placement of teachers, except in rare instances; this prerogative is reserved by school officials for themselves. And it is well that they do so, for the Committee does not seek or desire this privilege, as it entails too much responsibility. Again, vacancies which are reported to this Committee are reported in most instances to all of the educational institutions of the state and nominations for the vacancies are requested. At each institution, of course, there are prospective teachers of mathematics registered with the Placement Bureau or Committee, and this, consequently, increases the number of applicants available for the positions open. To illustrate: whereas, this Committee had only 34 teachers seeking mathematics positions in 1923–24 and there were reported 148 vacancies, it is safe to say that at least one hundred of these vacancies were reported to more than one of the committees and the sum total of the applicants available at all of the institutions to which the appeal was sent might easily reach one hundred or more. This is one difficulty of placement which the casual reader of figures does not see.

Another difficulty of placement is to be found in the combinations sought—e.g., mathematics and science, mathematics and Spanish, mathematics and athletics, etc. It is unfortunately a fact that many students of the University who go into teaching do not fit themselves for sponsoring any of the extra-curricular activities; they are content to prepare themselves for classroom teaching only in the subject which makes the greatest appeal to them. The day of such teachers is almost passed. It has seemed to the secretary, on limited knowledge, that students in other institutions have more regard for these requirements than our own students here; however, this is merely an impression and data enough for a conclusive statement in the matter are not available.

Time does not allow a compilation of figures showing the total number of those registering who are successful in receiving places. This would, of course, be an interesting complement to the comparison just presented here between the demand and the supply.

Another interesting study would be the training of those applying for positions as compared with the training of those teaching the subject. Too often it is the case that those accepting positions in mathematics are not the ones best prepared from an academic standpoint for the positions.

Still another interesting study would be a comparison of the figures showing the demand and supply and placement of teachers of mathematics in colleges and institutions of higher learning. They might easily be compiled from our records by one taking the time to do so.

A SHORTAGE OF MATHEMATICS TEACHERS

B. H. MILLER

Chairman, Commission on the Curriculum of T.S.T.A.

Dear Miss Decherd:

Here are some tables which indicate the shortage of teachers with suitable training in mathematics for high-school places.

You will note that 1,715 people are teaching high-school mathematics; that only 621 majored in mathematics; and that of these only 510 are teaching mathematics. (This number is limited to those who hold degrees. A total of 605 reported mathematics as a major as you will see by Table I. But I did not know that these people had really done enough work to be entitled to recognition as having completed a major in mathematics.)

It is interesting to note that there are almost exactly as many majors in English as there are teaching places in English; but that only 1,096 out of 1,876 of them are teaching English.

There is an amazing amount of assignment of teachers in fields outside of their major and minor studies while in college. School officials could probably do a great deal to remedy this trouble by reassigning their teachers in keeping with their major college training.

But in order to secure any adequate and permanent relief it will be necessary for the colleges to re-direct the selection of majors and minors. You can see by Table II that the distribution of major and minor subjects of college seniors in Texas colleges for 1930–1931 is far from the distribution of teaching assignments in high schools.

According to this table which is taken from reports received from four-year colleges in Texas concerning the major studies of seniors in the class of 1931 it is easy to see that with 580 majors in English and 83 majors in mathematics there is not a proper balance or distribution, and that mathematics seems to go begging.

Table II is not a complete record of all seniors; but contains all that had been received up to the time it was compiled. Later a revised table will be issued which will contain more names, but which will vary only a little from the percentage distribution shown above.

I have learned recently that there are some people who seem to think that it is not necessary to assign teachers according to their training and major interests. Probably that is one reason why primary teachers are trying to get into high school while some high-school teachers are trying to be primary teachers.

Column three of Table I could be used as a fair index of a proper distribution of majors for prospective high-school teachers in Texas. It would be about this:

	%		%
Athletics	9.2	Latin	4.2
Commercial	5.3	Mathematics	18.9
Economics	1.2	Science	11.0
English	20.6	Spanish	9.0
French and German	0.7	Vocational	13.1
History	17.2	Scattering	3.6
Home Economics	5.2		

It is interesting to compare this table with the actual distribution shown in Table III.

I shall be able to publish some more interesting and complete data some time later.

TABLE I. Distribution of High-school Teachers as to Subjects Taught, Major Subjects in College, and Collegiate Training

Subject Taught	Major Study Same as Subject Taught	Major Study Other Than Subject Taught	Total Number Who Teach This Subject
Athletics	7	77	84
Commercial	177	312	489
Economics	15	93	108
English	1,300	577	1,877
French	22	26	48
German	2	11	13
History	760	815	1,575
Home Economics	433	44	477
Latin	154	231	385
Mathematics	605	1,110	1,715
Music	60	37	97
Public Speaking	30	65	95
Physical Training	40	91	131
Science	471	572	1,043
Spanish	378	441	819
Vocational	192	89	281
Totals	4,646	4,591	9,237

TABLE II. Majors in Principal Large Fields, College Seniors, Texas, 1930–31

		%
Education	124	7.0
English	580	34.0
History and Government	420	24.7
Foreign Languages	98	6.0
Mathematics	83	5.0
Sciences	189	11.1
Vocational	201	11.8
Totals	1,695	100.0

TABLE III. Distribution of High-school Teachers Whose Major Study in College Was Mathematics as to Teaching Assignments*

Subject Taught	Number Teaching This Subject	Per Cent Teaching This Subject
Mathematics	510	82.1
Science	36	5.8
History	17	2.8
Commercial	13	2.1
English	17	1.8
Other subjects	34	5.4
Total	621	100.0

*Data from Bulletin No. 248, State Department of Education, Texas.

THE THREE FAMOUS PROBLEMS OF GREEK GEOMETRY

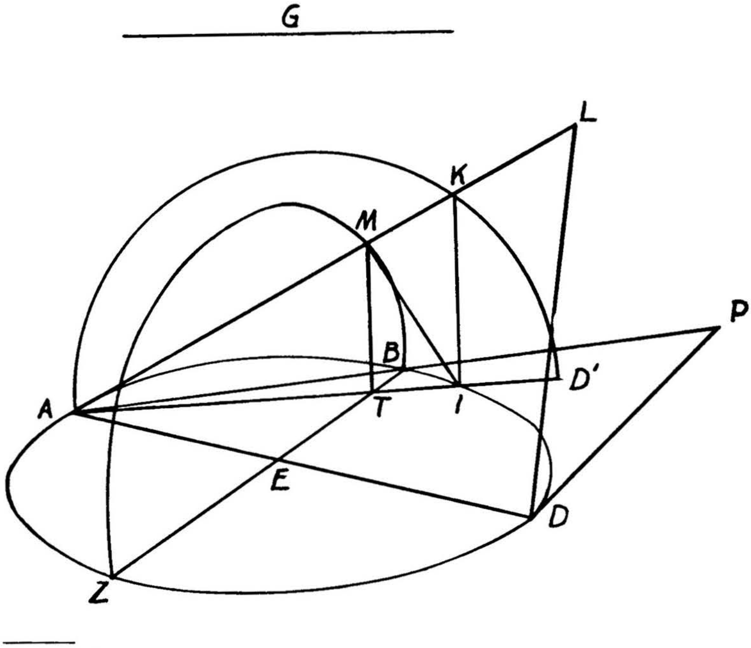
CHAPTER IV

CONSTRUCTION BY MEANS OF HIGHER CURVES[1]

MRS. DELLA HOUSSELS

Austin, Texas

Archytas (about 400 B.C.) was one of the first Greek mathematicians to give a solution to the duplication problem or to construct two mean proportionals to two given straight lines. The following is called the invention of Archytas as Eudemus relates it:



[1]The first two chapters and the first part of the third chapter of Mrs. Houssels' paper were printed in the last two numbers of the *Bulletin*. The latter part of the third chapter contains a long proof of the transcendence of e and π; this proof is taken with only slight modification from Veblen and Lennes: *Introduction to Infinitesimal Analysis*, pp. 19–29.

Let there be given two lines G and AD; it is required to find two mean proportionals to them. Let a circle $ABDZ$ be described round the greater line AD; and let the line AB, equal G, be inserted in it; and being produced let it meet at the point P, the line touching the circle at the point D. Further, let BEZ be drawn parallel to PD.

Now, let it be conceived that a semi-cylinder is erected on the semicircle ABD at right angles to it; also, at right angles to it, let there be drawn on the line AD a semicircle lying in the parallelogram of the cylinder. Then let this semicircle be turned from the point D toward B, the extremity A of the diameter remaining fixed; it will, in its circuit, cut the cylindrical surface and describe on it a certain line.

Again, if, the line AD remaining fixed, the triangle APD be turned round, with a motion contrary to that of the semicircle, it will form a conical surface with the straight line AP, which in its circuit will meet the cylindrical line (i.e., the line which is described on the cylindrical surface by the motion of the semicircle) in some point; and at the same time the point B will describe a semicircle on the surface of the cone.

Now, at the place of meeting of the lines, let the semicircle in the course of its motion have a position $D'KA$, and the triangle in the course of its opposite motion a position DLA; and let the point of said meeting be K.

Also let the semicircle described by B, be BMZ, and the common section of it and of the circle $BDZA$ be BZ; from the point K let a perpendicular be drawn to the plane of the semicircle BDA; it will fall on the periphery of the circle, because the cylinder stands perpendicularly. Let it fall, and let it be KI; and let the line joining the points I and A meet the line BZ in the point T; and let the right line AL meet the semicircle BMZ in the point M; and also let the lines KD', MI, MT be drawn.

Since, then, each of the semicircles $D'KA$, BMZ, is at right angles to the underlying plane, and, therefore, their common section MT is at right angles to the plane of the circle; so also is the line MT at right angles to BZ.

Therefore, the rectangle under the lines TB, TZ, that is under TA, TI, is equal to the square on MT.

The triangle AMI is therefore similar to each of the triangles MIT, MAT; and the angle IMA is right.

But the angle $D'KA$ is also right. Therefore, the lines KD', MI are parallel. And there will be the proportion:

As the line $D'A$ is to AK, i.e., KA to AI, so is the line IA to AM, on account of the similarity of the triangles. The four straight lines $D'A$, AK, AI, AM are, therefore, in continued proportion.

Also the line AM is equal to G, since it is equal to the line AB.

So the two lines AD, G being given, two mean proportionals have been found, viz. AK, AI.

Thus runs the solution of the Delian problem by Archytas. If AD be taken equal to $2G$, we have

$$\overline{AI}^3 = 2\overline{AM}^3 = 2G^3$$

The writer of a modern textbook might state the proportions more briefly, and might give the reasons for them more clearly; but we prefer to retain, as nearly as possible, the language of Archytas, which all will understand.[2]

Manaechmus (about 375 B.C.) furnished two most elegant solutions to the duplication problem. His solutions have been preserved to us by Eutocius in his "Commentary on the Second Book of the Treatise of Archimedes on the Sphere and Cylinder."

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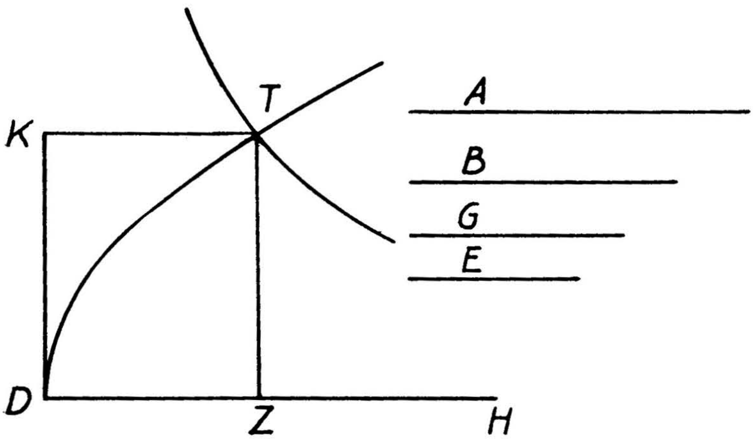
[2]Rupert, W. W.: *Famous Geometrical Theorems and Problems with Their History*, pp. 80–83.



First solution.—Let the two given lines be A, E and let the line DH be given in position and terminated at D; through D let a parabola be described whose axis is DH and parameter A. And let the squares of the ordinates drawn at right angles to DH be equal to the rectangles applied to A, and having for breadths the lines cut off by them to the point D. Let the parabola be described, and let it be DT, and let the line DK be drawn and let it be a perpendicular; and with the straight lines KD, DZ, as asymptotes, let the hyperbola be described, so that the lines drawn from it parallel to the lines KD and DZ shall form an area equal to the rectangle under A, E; the hyperbola will cut the parabola; let them cut in T, and let perpendiculars TK and TZ be drawn.

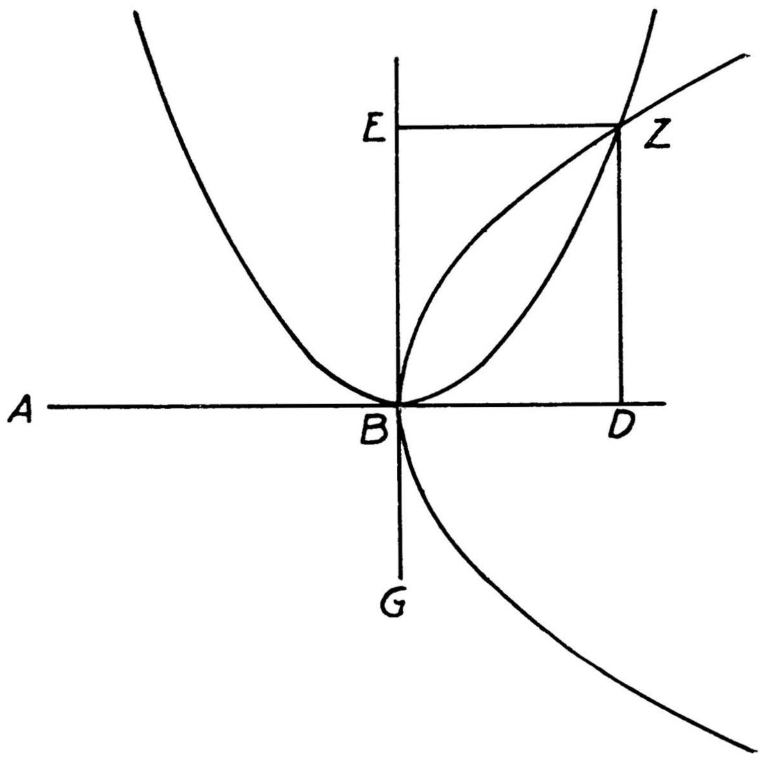
Since, then, the square on ZT is equal to the rectangle under A and DZ, there will be: as the line A is to ZT, so is the line ZT to ZD. $(A:ZT::ZT:ZD)$.

Again, since the rectangle under A and E is equal to the rectangle TZD, there will be: as the line A is to the line ZT, so is the line ZD to the line E $(A:ZT::ZD:E)$.

But the line A is to the line ZT, as the line ZT is to $ZD (A:ZT::ZT:ZD)$ and, therefore, as the line A is to the line *ZT*, so is the line *ZT* to *ZD*, and the line *ZD* to *E* ($A:ZT::ZT:ZD::ZD:E$).

Let the line *B* be taken equal to the line *ZT*, and the line *G* equal to the line *DZ*.

There will be, therefore, as the line *A* is to the line *B*, so is the line *B* to the line *G*, and the line *G* to *E*. The lines *A*, *B*, *G*, *E*, are therefore, in continual proportion.



Second solution.—Let *AB*, *BG* be the two given lines placed at right angles to each other, and let them be produced indefinitely from the point *B*; and let there be described about the axis *BE* a parabola, so that the square on any ordinate, ZE, shall be equal to the rectangle applied to the line BG with the line BE as height.

Again, let a parabola be described about DB as axis, so that the squares on its ordinates shall be equal to rectangles applied on the line AB. These parabolas cut each other; let them cut at the point Z, and from Z let perpendiculars ZD, ZE be drawn.

Since, then, in the parabola, the line ZE, that is the line DB, has been drawn, there will be: the rectangle under GB, BE equals the square on BD.

There is, therefore, as the line GB is to BD, so is the line DB to BE.

Again, since in the parabola the line ZD, that is, the line EB, has been drawn there will be: the rectangle under DB, BA equals the square on EB; there is, therefore, as the line DB is to BE, so is the line BE to BA.

But there was, as the line DB is to BE, so is the line GB to BD.

And thus there will be, therefore, as the line GB is to BD, so is the line DB to BE, and the line BE to BA.[3]

One of the best known of the ancient solutions of this most interesting problem was that of Diocles (about 180 B.C.), who used a curve called the *cissoid* which he had invented. He also invented a machine for tracing this curve.

To a circle draw a tangent (in the figure the vertical tangent on the right) and the diameter perpendicular to it.

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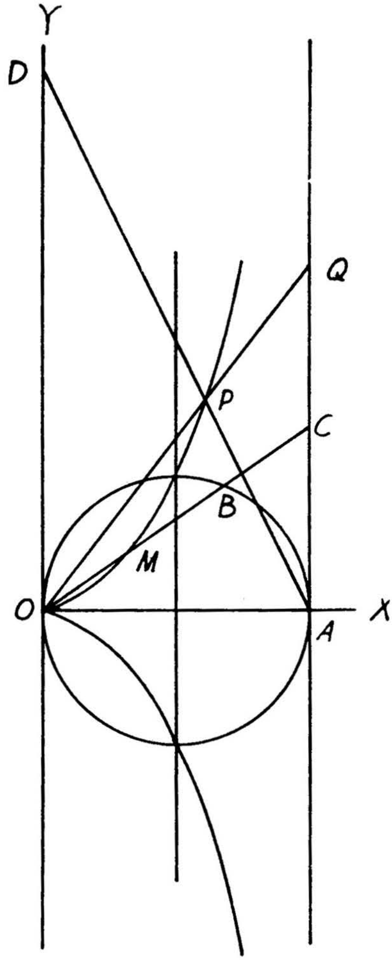
[3]Rupert, W. W.: *Famous Geometrical Theorems and Problems with Their History*, pp. 85–90.



Draw lines from O, the vertex of the circle thus determined, to points upon the tangent, and lay off from O upon each the segment lying between its intersection with the circle and the tangent. The locus of points so determined is the cissoid.

To derive the equation, let r be the radius vector, θ the angle it makes with the axis of X. If we produce r to the tangent on the right, and call the diameter of the circle unity, the total segment equals $1/\cos\theta$. The portion cut off by the circle is $\cos\theta$. The difference of the two segments is r, and

$$r = \frac{1}{\cos\theta} - \cos\theta = \frac{\sin^2\theta}{\cos\theta}$$

By transformation of co-ordinates, we obtain the Cartesian equation $(x^2 + y^2)x - y^2 = 0$.

This curve is of the third order, has a cusp at the origin, and is symmetric to the axis of X. The vertical tangent to the circle with which we began our construction is an asymptote. Finally the cissoid cuts the line at infinity in the circular points.

To show how to solve the Delian problem by the use of this curve, we write its equation in the following form:

$$(y/x)^3 = y/(1 - x)$$

We now construct the straight line $y/x = \lambda$. This cuts off upon the tangent $x = 1$ the segment λ, and intersects the cissoid in a point for which $y/(1 - x) = \lambda^3$. This is the equation of a straight line passing through the point $y = 0$, $x = 1$, and hence of the line joining this point to the point of the cissoid.

This line cuts off upon the axis of Y the intercept λ^3.

We now see how $\sqrt[3]{2}$ may be constructed. Lay off upon the axis of Y the intercept 2, join this point to the point $x = 1$, $y = 0$, and through its intersection with the cissoid draw a line from the origin to the tangent $x = 1$. The intercept on this tangent equals $\sqrt[3]{2}$.[4]

One of the best known methods of trisecting an angle is that used by Nicomedes (about 180 B.C.), who invented a curve, the *conchoid*, which is constructed in the following manner:

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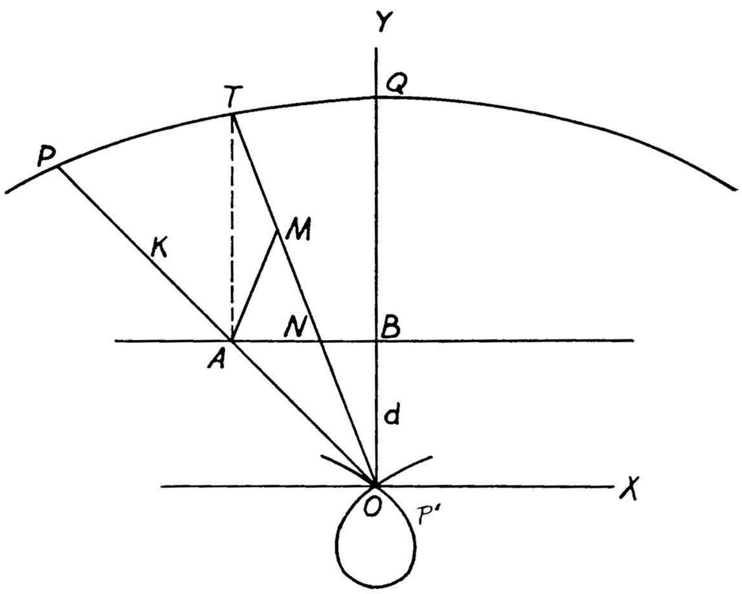
[4]Klein, F.: *Famous Problems in Elementary Geometry*, pp. 44–45.

Take a fixed point O which is d distant from a line AB and draw OX parallel to AB and OY perpendicular to OX. Then take any line AO through O, and on AO produced lay off $AP = AP' = K$, a constant. Then the locus of the points P and P' is the conchoid. According as K is greater than, equal to, or less than d we have a node, a cusp, or a conjugate point. The equation of the curve is

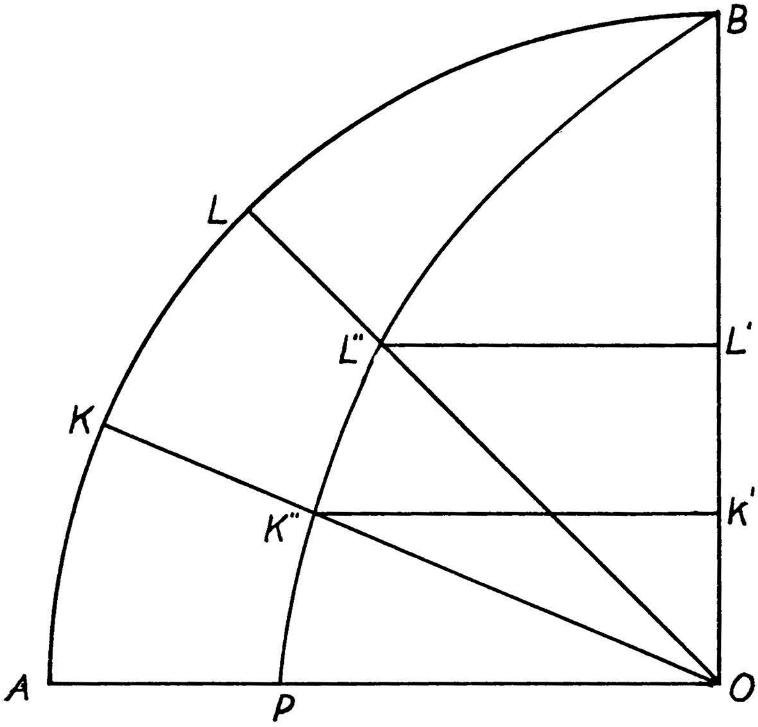
$$(x^2 + y^2)(x - d)^2 - k^2x^2 = 0.$$



In order to trisect an angle, let YOA be the angle to be trisected. From A construct AB perpendicular to OY. From O as a pole, with AB as a fixed straight line and $2AO$ as a constant distance, describe a conchoid to meet AO produced at P and to cut OY at Q. At A construct a perpendicular to AB meeting the curve at T. Draw OT and let it cut AB at N. Let M be the midpoint of NT. Then $MT = MN = MA$. But $NT = 2OA$ by construction of the conchoid.
Hence, $MA = OA$.

Hence, $AOM = AMO = 2\,ATM = 2\,TOQ$.
That is, $AOM = 2/3\ YOA$, and $TOQ = 1/3\ YOA$.[5]

Hippias of Elis invented a curve, the *quadratix*, by means of which he trisected an angle.



This curve is described as follows: Upon a quadrant of a circle cut off by two perpendicular radii, OA, OB, lie the points $A, \ldots K, L, \ldots B$. The radius $r = OA$ revolves with uniform velocity about O from the position OA to the position OB. At the same time a straight line g, always parallel to OA, moves with uniform velocity from the position OA to that of a tangent to the circle at B. If K' is the intersection of g with

[5]Smith, D. E.: *History of Mathematics*, pp. 299–300.

OB at the time when the moving radius falls upon OK, then the parallel to OA through K' meets the radius OK at a point K'' belonging to the quadratrix. If P is the intersection of OA with the quadratrix, it follows in part directly and in part by simple considerations, that $\frac{\text{arc } AK}{\text{arc } AL} = \frac{OK'}{OL'}$, a relation which solves any problem of angle sections.

Furthermore,

$$OP = \frac{2r}{\pi} \text{ or } \frac{OP}{OA} = \frac{OA}{\text{arc } AB},$$

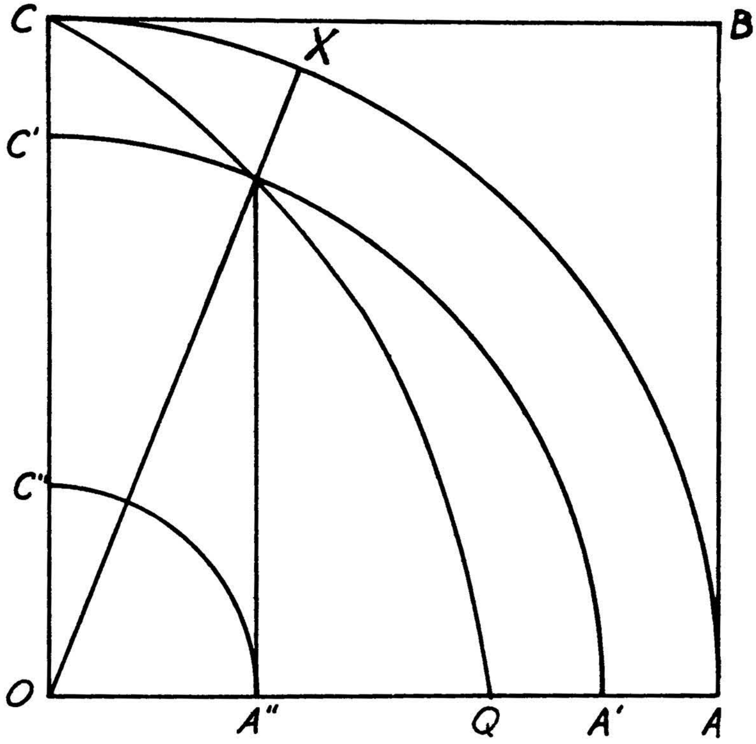
whence it is obvious that the quadrature of the circle depends upon the ratio in which the radius OA is divided by the point P of the quadratrix. If this ratio could be constructed by elementary geometry, the quadrature of the circle would be effected. The equation of the quadratrix in polar co-ordinates is

$$r = \frac{2\phi}{\pi} \cdot \frac{a}{\sin \phi},$$

where $a = OA$. Putting $\phi = o$, $r = r_o$, we have $\pi = \frac{2a}{r_o}$.[6]

It appears that the quadratrix was first invented for the trisection of an angle and that its relation to the quadrature of the circle was discovered later, as is shown by Dinostratus.

[6]Fink, Carl: *A Brief History of Mathematics*, translated by Beman and Smith, pp. 196–197.



In the figure it can be shown that $\frac{\text{arc } CXA}{CO} = \frac{CO}{OQ}$,

and since these terms are all straight lines except the quadrant CXA, it is possible to construct a straight line equal in length to the quadrant, and hence to rectify the circumference.[7]

The method Pappus used to proved the above statement was to let arc $CXA/CO > CO/OQ$ and then

$$\text{arc } CXA/CO < CO/OQ$$

and then prove that each inequality led to an absurdity.

We shall now conclude this discussion with a solution of the quadrature of the circle by means of the integraph.

[7] Smith, D. E.: *History of Mathematics*, pp. 305–306.

The Integraph and the Geometric Construction of π

Lindemann's theorem demonstrates the transcendence of π, and thus is shown the impossibility of solving the old problem of the quadrature of the circle, not only in the sense understood by the ancients but in a far more general manner. It is not only impossible to construct π with straight edge and compasses, but there is not even a curve of higher order defined by an integral algebraic equation for which π is the ordinate corresponding to a rational value of the abscissa. An actual construction of π can, then, be effected only by the aid of a transcendental curve. If such a construction is desired, we must use besides straight edge and compasses a "transcendental" apparatus which shall trace the curve by continuous motion.

Such an apparatus is the *integraph,* recently invented and described by a Russian engineer, Abdank-Abakanowicz, and constructed by Coradi of Zürich.

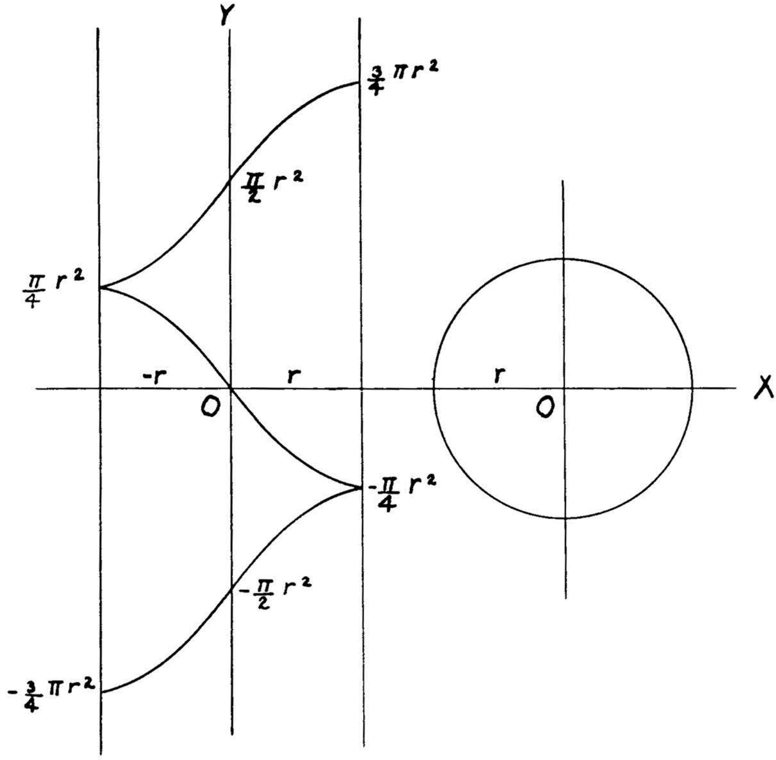
This instrument enables us to trace the *integral curve*

$$Y = F(x) = \int f(x)dx$$

when we have the *differential curve,* $y = f(x)$. For this purpose, we move the link work of the integraph so that the *guiding point* follows the differential curve; the *tracing point* will then trace the integral curve . . .

We shall simply indicate the principles of its working. For any point (x, y) of the differential curve construct the auxiliary triangle having for vertices the points (x, y), $(x, 0)$, $(x - 1, 0)$; the hypotenuse of this right-angled triangle makes with the axis of X an angle whose tangent $= y$.

Hence *this hypotenuse is parallel to the tangent to the integral curve at the point* (X, Y) *corresponding to the point* (x, y).



The apparatus should be so constructed that the tracing point shall move parallel to the variable direction of this hypotenuse, while the guiding point describes the differential curve. This is effected by connecting the tracing point with a sharp-edged roller whose plane is vertical and moves so as to be always parallel to this hypotenuse. A weight presses this roller firmly upon the paper so that its point of contact can advance only in the plane of the roller.

The practical object of the integraph is the approximate evaluation of definite integrals; for us its application to the construction of π is of especial interest.

Take for differential curve the circle

$$x^2 + y^2 = r^2;$$

the integral curve is then

$$Y = \int \sqrt{r^2 - x^2}\, dx = \frac{r^2}{2} \sin^{-1} \frac{x}{r} + \frac{x}{2} \sqrt{r^2 - x^2}$$

This curve consists of a series of congruent branches. The points where it meets the axis of Y have for ordinates

$$0, \pm \frac{r^2\pi}{2}, \ldots$$

Upon the lines $X = \pm r$ the intersections have for ordinates

$$r^2 \frac{\pi}{4},\ r^2 \frac{3\pi}{4}, \ldots$$

If we make $r = 1$, the ordinates of these intersections will determine the number π or its multiples.

It is worthy of notice that our apparatus enables us to trace the curve not in a tedious and inaccurate manner, but with ease and sharpness, especially if we use a tracing pen instead of a pencil.

Thus we have an actual constructive quadrature of the circle along the lines laid down by the ancients, for our curve is only a modification of the quadratrix considered by them.[8]

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[8]Klein, F.: *Famous Problems of Elementary Geometry*, translated by Beman and Smith, pp. 78–80.

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